Least Upper Bound of a Set

Definition 1.4.1. A subset E of $\mathbb{R}$ is **bounded above** if there exists $\beta \in \mathbb{R}$ such that $x \leq \beta$ for every $x \in E$. Such a β is called an **upper bound** of E.

Definition 1.4.3. Let E be a nonempty subset of $\mathbb{R}$ that is bounded above. An element $\alpha \in \mathbb{R}$ is called the **least upper bound** or **supremum** of E if

i) α is an upper bound of E, and

ii) if $\beta \in \mathbb{R}$ satisfies $\beta < \alpha$, then β is not an upper bound of E.

Least Upper Bound Property of $\mathbb{R}$

Property 1.4.6. Least Upper Bound Property of $\mathbb{R}$ Ever nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

Property 1.4.7. Greatest Lower Bound Property of $\mathbb{R}$ Ever nonempty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.

Consequences of the Least Upper Bound Property

Theorem 1.5.1. (Archimedian Property) If $x, y \in \mathbb{R}$ and $x > 0$, then there exists a positive integer n such that

$$nx > y.$$

Theorem 1.5.2. ($\mathbb{Q}$ is *dense* in $\mathbb{R}$) If $x, y \in \mathbb{R}$ and $x < y$, then there exists $r \in \mathbb{Q}$ such that

$$x < r < y.$$

Theorem 1.5.3. For every real number $x > 0$ and every positive integer n, there exists a unique positive real number y so that $y^n = x$.

Corollary 1.5.4. If a, b are positive real numbers, and n is a positive integer, then

$$(ab)^{1/n} = a^{1/n} b^{1/n}.$$

Countable and Uncountable Sets

Definition 1.7.1. Two sets A and B are said to be **equivalent** (or to have the same **cardinality**), denoted A B if there exists a *one-to-one* function of A *onto* B.

Definition 1.7.2. For each positive integer n, let $\mathbb{N}_n = \{1, 2, \ldots, n\}$. If A is a set, we say:

a) A is **finite** if A $\mathbb{N}_n$ for some n, or if $A = \emptyset$.
b) A is **infinite** if A is not finite.
c) A is **countable** if A $\mathbb{N}$.
d) A is **uncountable** if A is neither finite nor countable.
e) A is **at most countable** if A is finite or countable.

Theorem 1.7.4. $\mathbb{N} \times \mathbb{N}$ is countable.

Sequences

Definition 1.7.5. If A is a set, by a **sequence** in A we mean a function f from $\mathbb{N}$ into A. For each $n \in \mathbb{N}$, let $x_n = f(n)$. Then x_n is called the nth **term** of the sequence f.

Definition 1.7.6. Every infinite subset of a countable set is countable.

Theorem 1.7.7. If f maps $\mathbb{N}$ onto a, then A is at most countable.

Convergent Sequences

Definition 2.1.1. For a real number x, the **absolute value** of x is

$$|x| = \begin{cases} x, & \text{if } x > 0, \\ -x, & \text{if } x \leq 0. \end{cases}$$

Theorem 2.1.2.

a) $|-x| = |x|$ for all $x \in \mathbb{R}$;
b) $|xy|$=$|x||y|$ for all $x, y \in \mathbb{R}$;
c) $|x| = \sqrt{x^2}$ for all $x \in \mathbb{R}$;
d) if $r > 0$, then $|x| < r$ iff $-r < x < r$; and,
e) $-|x| \leq x \leq |x|$ for all $x \in \mathbb{R}$.

Theorem 2.1.3. (Triangle Inequality) For all $x, y \in \mathbb{R}$,

$$|x + y| \leq |x| + |y|.$$

Corollary 2.1.4. For all $x, y, z \in \mathbb{R}$,

a) $|x - y| \leq |x - z| + |z - y|$; and,
b) $||x| - |y|| \leq |x - y|$.

Definition 2.1.6. Let $p \in \mathbb{R}$ and let $\epsilon > 0$. The *ϵ-neighborhood* of the point p is

$$N_\epsilon(p) = \{x \in \mathbb{R} \mid |x - p| < \epsilon\}.$$

Definition 2.1.7. A sequence $\{p_n\}$ in $\mathbb{R}$ is said to **converge** if there exists a point $p \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists a positive integer n_0 such that $p_n \in N_\epsilon(p)$ for all $n \geq n_0$. If this is the case, we say that $\{p_n\}$ **converges** to p, or that p is the **limit** of the sequence $\{p_n\}$, and we write

$$\lim_{n\to\infty} p_n = p \qquad \text{or} \qquad p_n \to p.$$

Definition 2.1.9. A sequence $\{p_n\}$ in $\mathbb{R}$ is said to be **bounded** if there exists a positive constant M such that $|p_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 2.1.10.

a) If a sequence $\{p_n\}$ in $\mathbb{R}$ converges, then its limit is unique.
b) Every convergent sequence in $\mathbb{R}$ is bounded.

Limit Theorems

Theorem 2.2.1. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences of real numbers with

$$\lim_{n\to\infty} a_n = a \qquad \text{and} \qquad \lim_{n\to\infty} b_n = b,$$

then

a) $\lim_{n\to\infty} a_n + b_n = a + b$, and
b) $\lim_{n\to\infty} a_n b_n = ab$.
c) Furthermore, if $a \neq 0$, and $a_n \neq 0$ for all n, then $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{b}{a}$.

Corollary 2.2.2. If $\{a_n\}$ is a convergent sequence of real numbers with $\lim_{n\to\infty} a_n = a$, then for any $c \in \mathbb{R}$,

a) $\lim_{n\to\infty} a_n + c = a + c$, and
b) $\lim_{n\to\infty} c a_n = ca$.

Theorem 2.2.3. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. If $\{b_n\}$ is bounded and $\lim_{n\to\infty} a_n = 0$, then

$$\lim_{n\to\infty} a_n b_n = 0.$$

Theorem 2.2.4. Suppose $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers for which there exists $n_0 \in \mathbb{N}$ such that

$$a_n \leq b_n \leq c_n \qquad \text{for all } n \in \mathbb{N}, n \leq n_0,$$

and that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$. Then the sequence $\{b_n\}$ converges and

$$\lim_{n\to\infty} b_n = L.$$

Some Special Sequences

Theorem 2.2.5. (Binomial Theorem) For $a \in \mathbb{R}, b \in \mathbb{N}$,

$$(1+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k = \binom{n}{0} + \binom{n}{1} a + \binom{n}{2} a^2 + \ldots + \binom{n}{n} a^n$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is the binomial coefficient.

Theorem 2.2.6.

a) If $p > 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$.
b) If $p > 0$, then $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1$.
c) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.
d) If $p > 1$ and α is real, then $\lim\limits_{n\to\infty} \frac{n^\alpha}{p^n} = 0$.
e) If $|p| < 1$, then $\lim\limits_{n\to\infty} p^n = 0$.
f) For all $p \in \mathbb{R}$, $\lim\limits_{n\to\infty} \frac{p^n}{n!} = 0$.

Monotone Sequences

Definition 2.3.1. A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is said to be:

a) monotone increasing (or nondecreasing) if $a_n \leq b_n$ for all $n \in \mathbb{N}$;
b) monotone decreasing (or nonincreasing) if $a_n \geq b_n$ for all $n \in \mathbb{N}$;
c) monotone if it is either monotone increasing or monotone decreasing.

Theorem 2.3.2. If $\{a_n\}$ is monotone and bounded, then $\{a_n\}$ converges.

Corollary 2.3.3. (Nested Intervals Property) If $\{I_n\}$ is a sequence of closed and bounded intervals with $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$, then

$$\bigcap_{n=1}^{\infty} I_n = \emptyset.$$

Theorem 2.3.5. (Euler's Number) Euler's number, the base of the natural logarithm, and its powers can be expressed as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad \text{and} \quad e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

Definition 2.3.6. Let $\{a_n\}$ be a sequence of real numbers. We say that $\{a_n\}$ **apporaches infinity**, or that $\{a_n\}$ **diverges** to ∞, denoted $a_n \to \infty$, if for every positive real number M, there exists an integer $n_0 \in \mathbb{N}$ such that

$$a_n > M \qquad \text{for all} n \geq n_0.$$

Theorem 2.3.7. If $\{a_n\}$ is monotone increasing and not bounded above, then $a_n \to \infty$ as $n \to \infty$.

Subsequences and the Bolzano-Weierstrass Theorem

Definition 2.4.1. Given a sequence $\{p_n\}$ in $\mathbb{R}$, consider a sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers such that $n_1 < n_2 < n_2 < \ldots$. Then the sequence $\{p_n\}_{k=1}^{\infty}$ is called a **subsequence** of the sequence $\{p_n\}$.

Theorem 2.4.3. Let $\{p_n\}$ be a sequence in $\mathbb{R}$. If $\{p_n\}$ converges to p, then every subsequence of $\{p_n\}$ also converges to p.

Limit Point of a Set

Definition 2.4.5. Let E be4 a subset of $\mathbb{R}$.

a) A point $p \in \mathbb{R}$ is a **limit point** of E if every ϵ-neighborhood $N_\epsilon(p)$ of p contains a point $q \in E$ with $q \neq p$.

b) A point $p \in E$ that is not a limit point of E is called an **isolated point** of E.

Remarks. In the definition of limit point it is not required that p is a point of E. Also, a point $p \in E$ is an isolated point of E if there exists an $\epsilon > 0$ such that $N_\epsilon(p) \cap E = \{p\}$.

Theorem 2.4.7. Let E be a subset of $\mathbb{R}$.

a) If p is a limit point of E, then every neighborhood of p contains infinitely many points of E.

b) If p is a limit point of E, then there exists a sequene $\{p_n\}$ in E with $p_n \neq p$ for all $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} sp_n = p.$$

Corollary 2.4.8. A finite set has no limit points.

Balzano-Weierstrass Theorem

Theorem 2.4.10. (Balzano-Weierstrass) Every bounded infinite subset of $\mathbb{R}$ has a limit point.

Corollary 2.4.11. (Bolzano-Weierstrass) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Theorem 2.4.12. Let $\{p_n\}$ be a sequence in $\mathbb{R}$. If p is a limit point of $\{p_n \mid n \in \mathbb{N}\}$, then there exists a subsequence $\{p_n\}$ of $\{p\}$ such that $p_n \to p$ as $k \to \infty$.

Limit Superior and Inferior of a Sequence

Definition 2.5.1. Let $\{s_n\}$ be a sequence in $\mathbb{R}$. The **limit superior** of $\{s_n\}$, denoted $\overline{\lim} s_n$ is defined as

$$\overline{\lim_{n\to\infty}} s_n = \lim_{k\to\infty} b_k = \inf_{k\in\mathbb{N}} \sup\{s_n \mid n \leq k\}.$$

The **limit inferior** of $\{s_n\}$, denoted $\lim s_n$ is defined as

$$\underline{\lim_{n\to\infty}} s_n = \lim_{k\to\infty} a_k = \sup_{k\in\mathbb{N}} \inf\{s_n \mid n \geq k\}.$$

Theorem 2.5.3. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$.

a) Suppose $\overline{\lim_{n\to\infty}} s_n \in \mathbb{R}$. Then $\beta = \overline{\lim_{n\to\infty}} s_n$ if and only if for all $\epsilon > 0$,

i) there exists $n_0 \in \mathbb{N}$ such that $s_n < \beta + \epsilon$ for all $n \leq n_0$, and

ii) given $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ with $k \geq n$ such that $s + k > \beta - \epsilon$.

b) $\overline{\lim_{n\to\infty}} s_n = \infty$ if and only if given M and $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ with $k \geq n$ such that $s_k \geq M$.

c) $\overline{\lim_{n\to\infty}} s_n = -\infty$ if and only if $s_n \to -\infty$ as $n \to \infty$.

Remark. The statement "$s_n < \beta + \epsilon$ for all $n \neq n_0$" means that $s_n < \beta + \epsilon$ for all but finitely many n. On the other hand, the statement "given n, there exists $k \in \mathbb{N}$ with $k \geq n$ such that $s_l > \beta - \epsilon$" means that $s_n > \beta - \epsilon$ for infinitely many indices n.

Theorem 2.5.4. Let $\{s_n\}$ be a sequence in $\mathbb{R}$.

a) Suppose $\underline{\lim}_{n\to\infty} s_n \in \mathbb{R}$. Then $\alpha = \underline{\lim}_{n\to\infty} s_n$ if and only if for all $\epsilon > 0$,

i) there exists $n_0 \in \mathbb{N}$ such that $s_n > \alpha - \epsilon$ for all $n \geq n_0$, and

ii) given $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ with $k \geq n$ such that $s + k < \alpha + \epsilon$.

b) $\underline{\lim}_{n\to\infty} s_n = -\infty$ if and only if given M and $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ with $k \geq n$ such that $s_k \leq M$.

c) $\underline{\lim}_{n\to\infty} s_n = \infty$ if and only if $s_n \to \infty$ as $n \to \infty$.

Corollary 2.5.5. $\overline{\lim}_{n\to\infty} s_n = \underline{\lim}_{n\to\infty} s_n$ if and only if $\lim_{n\to\infty} s_n$ exists in $\mathbb{R} \cup \{-\infty, \infty\}$.

Theorem 2.5.6. Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences in $\mathbb{R}$. Then

$$\underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n \leq \underline{\lim}_{n\to\infty} (a_n + b_n) \leq \underline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n. \leq \overline{\lim}_{n\to\infty} (a_n + b_n) \leq \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n.$$

Theorem 2.5.7. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, and let

$$E = \text{the set of subsequential limits of } s_n \text{ in } \mathbb{R} \cup \{-\infty, \infty\}.$$

Then $\overline{\lim}_{n\to\infty} s_n$ and $\underline{\lim}_{n\to\infty} s_n$ are in E and

a) $\overline{\lim}_{n\to\infty} s_n = \sup E$, and

b) $\underline{\lim}_{n\to\infty} s_n = \inf E$.

Cauchy Sequences

Definition 2.6.1. A sequence $\{p_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is a **Cauchy sequence** if for every $\epsilon > 0$, there exists a positive integer n_0 such that

$$|p_n - p_m| < \epsilon$$

for all integers $n, m \geq n_0$.

Remark. In Definition 2.6.1, the criterion $|p_n - p_m| < \epsilon$ for all integers $n, m \geq n_0$ is equivalent to

$$\lim_{n\to\infty} |p_{n+k} - p_n| = 0.$$

Theorem 2.6.2.

a) Every convergent sequence in $\mathbb{R}$ is a Cauchy sequence.

b) Every Cauchy sequence is bounded.

Theorem 2.6.3. If $\{p_n\}$ is a Cauchy sequence in $\mathbb{R}$ that has a convergent subsequence, then the sequence $\{p_n\}$ converges.

Theorem 2.6.4. Every Cauchy sequence of real numbers converges.

Remark. The statement "every Cauchy sequence in $\mathbb{R}$ converges" is often expressed by sayin that $\mathbb{R}$ is **complete**. Since the proof of Theorem 2.6.4 used the Bolzano-Weierstrass theorem, the *completeness of* $\mathbb{R}$ ultimately depends on the least upper bound property of $\mathbb{R}$. Conversely, if we assume completeness of $\mathbb{R}$, then we can prove that $\mathbb{R}$ satisfies the least upper bound property. For this reason, the least upper bound or supremum property of $\mathbb{R}$ is often called the **completeness property** of $\mathbb{R}$.

Series of Real Numbers

Definition 2.7.1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let $\{s_n\}_{n=1}^{\infty}$ be the sequence obtained from $\{a_n\}$, where for each $n \in \mathbb{N}$, $s_n = \sum_{k=1}^{n} a_k$. The sequence $\{s_n\}$ is called an **infinite series**, or **series**, and is denoted either as

$$\sum_{k=1}^{\infty} a_k \text{ or as } a_1 + a_2 + \ldots + a_n + \ldots.$$

For each $n \in \mathbb{N}$, s_n is called the nth **partial sum** of the series and a_n is called the nth **term** of the series.
The series $\sum_{k=1}^{\infty} a_k$ **converges** if and only if the sequence $\{s_n\}$ of nth partial sums converges in $\mathbb{R}$. If $\lim\limits_{n\to\infty} s_n = s$, then s is called the **sum** of the series, and we write

$$s = \sum_{k=1}^{\infty} a_k.$$

The Cauchy Criterion

Theorem 2.7.3. (Cauchy Criterion) The series $\sum_{k=1}^{\infty} a_k$ converges if and only if given $\epsilon > 0$, there exists a positive integer n_0, such that

$$\left| \sum_{k=n+1}^{m} a_k \right| < \epsilon$$

for all $m > n \leq n_0$.

Corollary 2.7.5. If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim\limits_{k\to\infty} a_k = 0$.

Theorem 2.7.6. Suppose $a_k \geq 0$ for all $k \in \mathbb{N}$. Then $\sum_{k=1}^{\infty}$ converges if and only if $\{s_n\}$ is bounded above.

Structure of Point Sets

Open and Closed Sets

Definition 3.1.1. Let E be a subset of $\mathbb{R}$. A point $p \in E$ is called an **interior point** of E if there exists an $\epsilon > 0$ such that $N_\epsilon(p) \subset E$. The set of interior points of E is denoted by $Int(E)$, and is called the **interior** of E.

Open and Closed Sets

Definition 3.1.4.

- **a)** A subset O of $\mathbb{R}$ is **open** if every point of O is an interior point of O.
- **b)** A subset F of $\mathbb{R}$ is **closed** if $F^c = \mathbb{R}/F$ is open.

Remarks. From the definition of an interior point it should be clear that a set $O \subset \mathbb{R}$ is open if and only if for every $p \in O$ there exists an $\epsilon > 0$ (depending on p) so that $N_\epsilon(p) \subset O$. In Theorem 3.1.9 we will provide a characterization of closed sets in terms of limit points.

Theorem 3.1.5. Every open interval of $\mathbb{R}$ is an open subset of $\mathbb{R}$.

Theorem 3.1.6.

- **a)** For any collection $\{O_\alpha\}_{\alpha\in A}$ of open subsets of $\mathbb{R} \bigcup_{\alpha\in A} O_\alpha$ is open.
- **b)** For any finite collection $\{O_1, \ldots, O_n\}$ of open subsets of $\mathbb{R}$, $\bigcap_{j=1}^{n} = O_j$ is open.

Theorem 3.1.7.

- **a)** For any collection $\{F_\alpha\}_{\alpha\in A}$ of closed subsets of $\mathbb{R} \bigcap_{\alpha\in A} F_\alpha$ is closed.
- **b)** For any finite collection $\{F_1, \ldots, F_n\}$ of closed subsets of $\mathbb{R}$, $\bigcup_{j=1}^{n} F_j$ is closed.

Theorem 3.1.9. A subset F of $\mathbb{R}$ is closed if and only if F contains all its limit points.

Closure of a Set

Definition 3.1.10. If E is a subset of $\mathbb{R}$, let E' denote the set of limit points of E. The

closure of E, denoted $\overline{E}$, is defined as

$$\overline{E} = E \cup E'.$$

Theorem 3.1.11. If E is a subset of $\mathbb{R}$, then

a) $\overline{E}$ is closed.

b) $E = \overline{E}$ if and only if E is closed.

c) $\overline{E} \subset F$ for every closed set $F \subset \mathbb{R}$ such that $E \subset F$.

Definition 3.1.12. A subset D of $\mathbb{R}$ is **dense** in $\mathbb{R}$ if $\overline{D} = \mathbb{R}$.

Characterization of the Open Subsets of $\mathbb{R}$

Theorem 3.1.13. If U is an open subset of $\mathbb{R}$, then there exists a finite or contable collection $\{I_n\}$ of pairwise disjoint open intervals such that

$$U = \bigcup_n I_n.$$

Relatively Open and Closed Sets

Definition 3.1.14. Let X be a subset of $\mathbb{R}$.

a) A subset U of X is **open in** (or open relative to) X if for every $o \in U$, there exists $\epsilon > 0$ such that $N_\epsilon(p) \cap X \subset U$.

b) A subset C of X is **closed in** (or closed relative to) X if X
C is open in X.

Theorem 3.1.16. Let X be a subset of $\mathbb{R}$.

a) A subset U of X is open in X if and only if $U = X \cap O$ for some open subset O of $\mathbb{R}$.

b) A subset C of X is closed in X if and only if $C = X \cap F$ for some closed subset F of $\mathbb{R}$.

Connected Sets

Definition 3.1.17. A subset A of $\mathbb{R}$ is **connected** if there do not exist two disjoint open sets U and V such that

a) $A \cap U \neq \phi$ ad $A \cap V \neq \phi$, and

b) $(A \cap U) \cup (A \cap V) = A$.

Theorem 3.1.18. A subset of $\mathbb{R}$ is connected if and only if it is an interval.

Compact Sets

Definition 3.2.1. Let $E \subset \mathbb{R}$. A *collection* $\{O_\alpha\}_{\alpha \in A}$ of open subsets of $\mathbb{R}$ is an **open cover** of E if

$$E \subset \bigcup_{\alpha \in A} O_\alpha.$$

Definition 3.2.3. A subset K of $\mathbb{R}$ is **compact** if every over cover of K has a finite subcover of K; that is, if $\{)_\alpha\}_{\alpha \in A}$ is an open cover of K, then there exists $\alpha_1, \ldots, \alpha_n \in A$ such that

$$K \subset \bigcup_{j=1}^{n} O_{\alpha_j}.$$

Remark. To prove (using the definition) that a given set K is compact, we must prove that for <u>every</u> open cover $\{O_alpha\}$ of K, there exists a finite subcollection $O_{\alpha_1}, \ldots, O_{\alpha_n}$ whose union covers K.

Properties of Compact Sets

Theorem 3.2.5.
  a) Every compact subset of $\mathbb{R}$ is closed and bounded.
  b) Every closed subset of a compact set is compact.

Theorem 3.2.6. If S is an infinite subset of a compact set K, then S has a limit point in K.

Theorem 3.2.7. If $\{K_n\}_{n=1}^{\infty}$ is a sequence of nonempty compact subsets of $\mathbb{R}$ with $K_n \supset K_{n+1}$ for all n, then

$$K = \bigcap_{n=1}^{\infty} K_n.$$

is nonempty and compact.

Characterization of the Compact Subsets of $\mathbb{R}$

Theorem 3.2.8. (Heine-Borel) Every closed and bounded interval $[a, b]$ is compact.

Theorem 3.2.9. (Heine-Borel-Bolzano-Weierstrass) Let K be a subset of $\mathbb{R}$. Then the following are equivalent:
  a) K is closed and bounded.

b) K is compact.
c) Every infinite subset of K has a limit point in K.

Theorem 3.2.10. Let K be a nonempty compact subset of $\mathbb{R}$. Then every sequence in K has a convergent subsequence that converges to a point in K.

Limits and Continuity

Definition 4.1.1. Let E be a subset of $\mathbb{R}$ and f a real-valued function with domain E. Suppose that p is a limit point of E. The function f has a **limit** at p if there exists a numer $L \in \mathbb{R}$ such that given any $\epsilon > 0$, there exists a $\delta > 0$ for which

$$|f(x) - L| < \epsilon$$

for all points $x \in E$ satisfying $0 < |x - p| < \delta$. If this is the case, we write

$$\lim_{x \to p} f(x) = L \text{ or } f(x) \to L \text{ as } x \to p.$$

Remarks

a) In the definition of limit, the choice of δ for a given ϵ may depend not only on ϵ and the function, but also on the point p.

b) If p is not a limit point of E, then for δ sufficiently small, there do not exist any $x \in E$ so that $0 < |x - p| < \delta$. Thus if p is an isolated point of E, the concept of the limit of a function at p has no meaning.

c) In the definition of limit, it is not required that $p \in E$, only that p is a limit point of E. Even if $p \in E$, and f has a limit at p, we may ver well have that

$$\lim_{x \to p} \neq f(p).$$

d) Let $E \subset \mathbb{R}$ and p a limit point of E. To show that a given function f does <u>*not*</u> have a limit at p, we must shiw that for every $L \in \mathbb{R}$, there exists an $\epsilon > 0$, such that for every $\delta > 0$, there exists an $x \in E$ with $0 < |x - p| < \delta$, for which

$$|f(x) - L| \geq \epsilon.$$

Sequential Criterion for Limits

Theorem 4.1.3. Let E be a subset of $\mathbb{R}$, p a limit point of E, and f a real-valued function defined on E. Then

$$\lim_{x \to p} f(x) = L \text{ if and only if } \lim_{n \to \infty} f(p_n) = L$$

for every sequence $\{p_n\}$ in e, with $p_n \neq p$ for all $n \in \mathbb{N}$ and $p_n \to p$.

Remark. Since p is a limit point of E, Theorem 2.4.7 guarantees the existence of a sequence $\{p_n\}$ in E with $p_n \neq p$ for all $n \in \mathbb{N}$ and $p_n \to p$.

Corollary 4.1.4. If f has a limit at p, then it is unique.

Limit Theorems

Theorem 4.1.6. Suppose $E \subset \mathbb{R}$, $f, g : E \to \mathbb{R}$ and p is a limit point of E. If

$$\lim_{x \to p} f(x) = A \text{ and } \lim_{x \to p} g(x) = B,$$

then

a) $\lim_{x \to p} [f(x) + g(x)] = A + B$,

b) $\lim_{x \to p} f(x)g(x) = AB$,

c) $\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{A}{B}$, provided $B \neq 0$.

Definition 4.1.7. A real-valued function f defined on a set E is **bounded** on E if there exists a constant M such that $|f(x)| \leq M$ for all $x \in E$.

Theorem 4.1.8. Suppose $E \subset \mathbb{R}$, p is a limit point of E, and f, g are real-valud functions on E. If g is bounded on E and $\lim_{n \to p} f(x) = 0$, then

$$\lim_{x \to p} f(x)g(x) = 0.$$

Theorem 4.1.9. Suppose $E \in \mathbb{R}$, p is a limit point of E, and f, g, h are functions from E into $\mathbb{R}$ satisfying

$$g(x) \leq f(x) \leq h(x) \text{ for all } x \in E.$$

If $\lim_{x \to p} g(x) = \lim_{x \to p} h(x) = L$, then $\lim_{x \to p} f(x) = L$.

Limits at Infinity

Definition 4.1.11. Let f be a real-valued function such that $\mathrm{Dom} f \cap (a, \infty) \neq \emptyset$ for every $a \in \mathbb{R}$. The function f has a **limit at** ∞ if there exists a number $L \in \mathbb{R}$ such that given $\epsilon > 0$, there exists a real number M for which

$$|f(x) - L| < \epsilon$$

for all $x \in \mathrm{Dom} f \cap (M, \infty)$. If this is the case, we write

$$\lim_{x \to \infty} f(x) = L.$$

Similarly, if $\mathrm{Dom} f \cap (-\infty, b) \neq \emptyset$ for every $b \in \mathbb{R}$,

$$\lim_{x \to -\infty} f(x) = L$$

if and only if given $\epsilon > 0$, there exists a real number M such that

$$|f(x) - L| < \epsilon$$

for all $x \in \mathrm{Dom} f \cap (-\infty, M)$.

Continuous Functions

Definition 4.2.1. Let E be a subset of $\mathbb{R}$ and f a real-valued function with domain E. The function f is **continuous at a point** $p \in E$, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(p)| < \epsilon$$

for all $x \in E$ with $|x - p| < \delta$. The function f is **continuous on** E if and only if f is continuous at every point $p \in E$.

Remarks.

a) If $p \in E$ is a limit point of E, then f is continous at p if and only if

$$\lim_{x \to p} f(x) = f(p).$$

Also, as a consequence of Theorem 4.1.3, f is continuous at p if and only if

$$\lim_{x \to \infty} f(p_n) = f(p)$$

for every sequence $\{p_n\}$ in E with $p_n \to p$.

b) If $p \in E$ is an isolated point, then <u>*every*</u> function f on E is continuous at p. This follows immediately from the fact that for an isolated point p of E, there exists a $\delta > 0$ such that $N_\delta(p) \cap E = \{p\}$.

Theorem 4.2.3. If $E \subset \mathbb{R}$ and $f, g : E \to \mathbb{R}$ are continuous at $p \in E$, then

a) $f + g$ and $f - g$ are continous are p, and

b) fg is continuous at p.

c) If $g(x) \neq 0$ for all $x \in E$, then f/g is continuous at p.

Composition of Continuous Functions

Theorem 4.2.4. Let $A, B \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions such that *Range* $f \subset B$. If f is continuous at $p \in A$ and g is continuous at $f(p)$, then $h = g \circ f$ is continuous at p.

Topological Characterization of Continuity

Theorem 4.2.6. Let E be a subset of $\mathbb{R}$ and let f be a real-valued function on E. Then f is continuous on E if and only if $f^{-1}(V)$ is open in E for every open subset V of $\mathbb{R}$.

Continuity and Compactness

Theorem 4.2.8. If K is a compact subset of $\mathbb{R}$ and if $f : K \to \mathbb{R}$ is continuous on K, then $f(K)$ is compact.

Corollary 4.2.9. Let K be a compact subset of $\mathbb{R}$ and let $f : K \to \mathbb{R}$ be continuous. There exists $p, q \in K$ such that

$$f(q) \leq f(x) \leq f(p) \text{ for all } x \in K.$$

Intermediate Value Theorem

Theorem 4.2.11. (Intermediate Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose $f(a) < f(b)$. If γ is a number satisfying

$$f(a) < \gamma < f(b),$$

then there exists $c \in (a, b)$ such that $f(c) = \gamma$.

Corollary 4.2.12. If $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is continuous on I, then $f(I)$ is an interval.

Corollary 4.2.13. For every real number $\gamma > 0$ and every positive integer n, there exists a unique positive real number y so that $y^n = \gamma$.

Corollary 4.2.14. If $f : [0, 1] \to [0, 1]$ is continuous, then there exists $y \in [0, 1]$ such that $f(y) = y$.