Chapter 14

8. Prove that the intersection of any set of ideals of a ring is an ideal.

16. If A and B are ideals of a commutative ring R with unity and $A + B = R$, show that $A \cap B = AB$.

40. Let a and b belong to a commutative ring R. Prove that $\{x \in R \mid ax \in bR\}$ is an ideal.

Chapter 15

22. Suppose ϕ is a ring homomorphism from $\mathbb{Z} \oplus \mathbb{Z}$ into $\mathbb{Z} \oplus \mathbb{Z}$. What are the possibilities for $\phi[(1, 0)]$?

46. Show that the only ring automorphism of the real numbers is the identity mapping.

50. Let $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ and $\mathbb{Q}[\sqrt{5}] = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$. Show that these two rings are not ring-isomorphic.

Chapter 16

8. Let R be a commutative ring. Show that $R[x]$ has a subring isomorphic to R.

12. Let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1$ in $\mathbb{Z}_7[x]$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

22. Prove that $\mathbb{Z}[x]$ is not a principal ideal domain. (Compare with Theorem 16.3.)

Chapter 17

4. Suppose that $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in \mathbb{Z}[x]$. If r is a rational and $x - r$ divides $f(x)$, show that r is an integer.

6. Suppose that $f(x) \in \mathbb{Z}_p[x]$ and is irreducible over $\mathbb{Z}_p$, where p is a prime. If def $f(x) = n$, prove that $\mathbb{Z}_p[x]/\langle f(x)\rangle$ is a field with p^n elements.

8. Construct a field over order 27.

10. Determine which of the polynomials below is (are) irreducible ove $\mathbb{Q}$ (with new polynomials).

20. Let $f(x) \in \mathbb{Z}_p[x]$. Prove that if $f(x)$ has no factor of the form $x^2 + ax + b$, then it has no quadratic factor over $\mathbb{Z}_p$.

24. Find all zeros of $f(x) = 3x^2 + x + 4$ over $\mathbb{Z}_7$ by substitution. Find all zeros of $f(x)$ by using the Quadratic Formula $(-b \pm \sqrt{b^2 - 4ac})(2a^-1)$ (all calculations are done in $\mathbb{Z}_7$). Do your answers agree? Should they? Find all zeros of $g(x) = 2x^2 + x + 3$ over $\mathbb{Z}_5$ by substitution. Try the Quadratic Formula on $g(x)$. Do your answers agree? State necessary and sufficient conditions for the Quadratic Formula to yield the zeros of a quadratic from $\mathbb{Z}_p[x]$, where p is a prime greater than 2.

32. Prove that the ideal $\left\langle x^2 + 1 \right\rangle$ is prime in $\mathbb{Z}[x]$ but not maximal in $\mathbb{Z}[x]$.

36. Show that the two-sided die labeled with 1 and 4 and another 18-sided die labeled with 1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 8 yield the same probabilities as an ordinary pair of cubes labeled 1 through 5. Carry out an analysis similar to that given in Example 12 to derine these labels.

Chapter 18

2. In an integral domain, show that a and b are associates if and only if $\langle a \rangle = \langle b \rangle$.

4. In an integral domain, show that the product of an irreducible and a unit is irreducible.

8. Let D be a Euclidean domain with measure d. Prove that u is a unit of D if and only if $d(u) = d(1)$.

12. Let D be a principal ideal domain. Show that every proper ideal of D is contained in a maximal ideal.

18. Prove that 7 is irreducible in $\mathbb{Z}[\sqrt{6}]$, even though $N(7)$ is not prime.

24. Let a and b be idempotents in a commutative ring. Show that eah of the following is also an idempotent: $ab, a - ab, a + b - ab, a + b - 2ab$.

26. If a and b belong to $\mathbb{Z}[\sqrt{d}]$, where d is not divisible by the square of a prime and ab is a unit, prove that a and b are units.

30. Show that $2x^2 + 4x + 3 \in \mathbb{Z}_5[x]$ factors as $(3x + 2)(x + 4)$ and $(4x + 1)(2x + 3)$. Explain why this does not contradict the corollary of Theorem 18.3.

Chapter 19

6. Determine whether or not the set

$$\left\{ \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

is linearly independent over $\mathbb{Z}_5$.

8. If $\{v_1, v_2, \ldots, v_n\}$ is a linearly dependent set of vectors, prove that one of these vectors is a linear combination of the others.

10. (Every independent set is contained in a basis). Let V be a finite-dimensional vector space and let $\{v_1, v_2, \ldots, v_n\}$ be a linearly independent subset of V. Show that there are vectors $w_1, w_2, \ldots, w_m$ such that $\{v_1, v_2, \ldots, v_n, w_1, \ldots, w_m\}$ is a basis for V.

14. Let $V = \mathbb{R}^3$ and $W = \{(a, b, c) \in V \mid a^2 + b^2 = c^2\}$. Is W a subspace of V? If so, what is its dimension?

16. Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c \in \mathbb{Q} \right\}$. Prove that V is a vector space over $\mathbb{Q}$ and find a basis for V over $\mathbb{Q}$.

24. Let U and W be subspaces of a vector space V. Show that $U \cap W$ is a subspace of V and that $U + W = \{u + w \mid u \in U, w \in W\}$ is a subspace of V.

28. Let T be a linear transformation of V onto W. Prove that the image of V under T is a subspace of W.

30. Let T be a linear transformation of V onto W. If $\{v_1, v_2, \ldots, v_n\}$ spans V, show that $\{T(v_1), T(v_1), \ldots, T(v_n)\}$ spans W.

Chapter 22

4. How many elements of the cyclic group $GF(81)*$ are generators?

6. Prove that the rings $\mathbb{Z}_3[x]/\left\langle x^2 + x + 2 \right\rangle$ and $\mathbb{Z}_3[x]/\left\langle x^2 + 2x + 2 \right\rangle$ are isomorphic.

14. If $f(x)$ is a cubic irreducible polynomial over $\mathbb{Z}_3$, prove that either x or $2x$ is a generator for the cylic group $\mathbb{Z}_3[x]/\langle f(x) \rangle *$.

16. Suppose that α and β belong to $GF(81)*$, with $|\alpha| = 5$ and $|\beta| = 16$. Show that $\alpha\beta$ is a generator of $GF(81)*$.